

LÉVY PROCESSES, MARTINGALES, REVERSED MARTINGALES AND ORTHOGONAL POLYNOMIALS

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ABSTRACT. We study class of Lévy processes having all moments. We define system of polynomial martingales $\{M_n(X_t, t), \mathcal{F}_{\leq t}\}_{n \geq 1}$, where $\mathcal{F}_{\leq t}$ is a suitable filtration defined below. We present some properties of these martingales. Among others we show that $M_1(X_t, t)/t$ is a reversed martingale and M_1 is a harness. We study also chances for martingales M_n multiplied by suitable function of t to be a reversed martingale. We show that for $n \geq 3$ it is possible only when the Lévy process in question is Gaussian (i.e. is a Wiener process). For $n = 2$ we present Lévy process different from Gaussian that has this property.

1. INTRODUCTION

There have been defined many families of polynomials associated or defined for Lévy process with all moments existing. The most popular ones are the Kaillath–Segall polynomials (see [14], [12], [15], [27]) connected with path’ structure of the process and the properties of multiple integrals of the process. There are also so called Teugels polynomials (see [16], [17]) associated with the properties of Lévy measure of the process.

Our approach is different. We are seeking polynomials functions $M_n(X_t, t)$ of process’s observations X_t at t that are martingales. We indicate conditions under which these polynomial multiplied by some deterministic functions of time parameter are reversed martingales or constitute a family of orthogonal, polynomial martingales. We also analyze the structure of so called ‘angular brackets’ of martingales M_n i.e. functions $p_n(t) = EM_n(X_t, t)^2$.

Of course there exists relation of our martingales to Kaillath–Segall polynomials (see [14] or Yablonski’s polynomials (see [27])). In 2011 in seminar presentations in Innsbruck J.L. Solé constructed polynomial martingales using Bell’s (or Yablonski’s) polynomials. We derive the form of these martingales once more, straightforwardly as illustration of the theory of stochastic process with polynomial regression as presented in [21]. We present many more properties of them then mentioned in Solé’s presentation. Of course on the way we point out relationship with Yablonski’s polynomials. The paper is organized as follows. The next Section 2 contains our main results. Section 3 contains some open problems that can be solved using technic presented in the paper and which we leave to more talented researchers. Finally Section 4 contains uninteresting or tedious proofs.

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Finally let us mention the fact that while analyzing consequences of the assumption that $\mu(t)M_2(X_t, t)$ is the reversed martingale we had to prove, believed to be new, interesting property of the so called tangent numbers (see (4.4)), numbers closely related to Bernoulli numbers.

2. POLYNOMIAL MARTINGALES

Let us formulate assumptions that will be in force throughout the paper. On the probability space (Ω, \mathcal{F}, P) there is defined Lévy stochastic process $\mathbf{X} = (X_t)_{t \geq 0}$, i.e. time homogeneous process with independent increments, continuous with probability. Let us define filtrations $\mathcal{F}_{\leq s} = \sigma(X_u : u \leq s)$ for $s > 0$, $\mathcal{F}_{\geq s} = \sigma(X_u : u \geq s)$ and $\mathcal{F}_{s,u} = \sigma(X_t : t \notin [s, u])$ if $s < t < u$.

Let us also stress that all equalities between random variables are understood to be with probability 1. Hence we drop abbreviation a.s. usually following equality between random variables for clarity of exposition.

We will be interested only in those Lévy processes which possess all moments. Such processes constitute a subclass of the class of all Lévy processes and the main tool of analysis here are moments functions. Hence we will not refer to Lévy measure that is the usual tool in analysis of Lévy processes. Instead we will use Kolmogorov's characterization of infinitely divisible distributions as presented e.g. in [11] to complement our analysis. Of course the two tools are closely related to one another since one can get all moment functions knowing characteristic function. We will use moment functions for the sake of completeness of the paper and also in order to illustrate the usage of recently obtained results of the paper [21].

Let us denote by $m_n(t)$ the n -th moment of the process i.e. $m_n(t) = EX_t^n$. We will assume that for all $n \geq 0$ functions $m_n(t)$ exist and are well defined.

We have the following set of easy observations.

Proposition 1. *i)*

$$(2.1) \quad m_n(s+t) = \sum_{j=0}^n \binom{n}{j} m_j(s) m_{n-j}(t),$$

for all $n \geq 0$ and $s, t \geq 0$.

ii) Let $Q(t; x) = \sum_{j \geq 0} m_j(t) x^j / j!$ be characteristic function of moment functions, then

$$Q(t; x) = \exp(tf(x)),$$

with $f(x) = \sum_{k \geq 1} c_k x^k / k!$ and $m_n(t) = \left. \frac{\partial^n \exp(tf(x))}{\partial x^n} \right|_{x=0}$ where coefficients c_i , $i = 1, \dots$ are such that for every $t \geq 0$ sequence $\{m_n(t)\}_{n \geq 0}$ is the moment sequence.

iii) $c_1 t = EX_t$, $\text{var}(X_t) = c_2 t$. Let $\hat{m}_n(t) = E(X_t - c_1 t)^n$ be the central moment sequence. Then $\sum_{j \geq 0} \hat{m}_n(t) x^j / j! = \exp(t(f(x) - c_1 x))$.

iv) Moment functions $m_n(t)$ satisfy the following set of differential equations: $\forall n > 0, t > 0$

$$(2.2) \quad m'_n(t) = \sum_{j=1}^n \binom{n}{j} c_j m_{n-j}(t).$$

v) Let $\{V_n(t)\}_{n \geq 0}$ be a family of lower triangular matrices composed from moments functions as follows: (i, j) -th entry of V_n i.e. $v_{i,j}(t)$ is equal to $\binom{i}{j} m_{i-j}(t)$.

We have:

$$(2.3) \quad V_n(t) = \exp(tC_n),$$

where matrix C_n is a lower triangular matrix with (i, j) entry equal to $\binom{i}{j} c_{i-j}$, where numbers c_i are defined in ii) and where we set $c_0 = 0$. Moreover for every $n > 0$ n polynomials defined by the formula

$$(2.4) \quad \mathcal{M}_n(t) = V_n(-t) \mathbf{X}_t^{(n)},$$

where $\mathbf{X}_t^{(n)} = [1, X_t, X_t^2, \dots, X_t^n]^T$ are martingales with respect to filtration $\mathcal{F}_{\leq t}$.

Proof. Is shifted to Section 4. \square

Remark 1. Coefficients c_i can be identified with the moments of the Kolmogorov's measure dK of the analyzed Lévy process. Recall that since we deal with the case of the process with finite variance we can use equivalent form of the Lévy canonical form of the infinitely divisible distribution namely the Kolmogorov's one (see e.g. [11], p.93, (10)). Applying this formula for $t = -ix$ we get

$$E \exp(xX_t) = \exp(tf(x)) = \exp(tc_1x + t \int_{-\infty}^{\infty} \frac{(\exp(xy) - 1 - xy)}{y^2} dK(y),$$

where $K(y)$ is a non-decreasing function with bounded variation such that $K(-\infty) = 0$ and $K(\infty) = \int_{\mathbb{R}} dK(y) = \text{var}(X_1) = c_2$. Confronting Kolmogorov's formula with the definition of coefficients c_i we arrive at the following interpretation of these parameters:

$$c_i = \int_{\mathbb{R}} y^{i-2} dK(y),$$

$i \geq 3$. Thus we see that coefficients c_i/c_2 are moments of probability measure $\frac{1}{c_2} dK(u)$ on the real line. In particular we deduce that

$$i) (c_4/c_2)^{1/2} \leq (c_6/c_2)^{1/4} \leq \dots \leq (c_{2k+2}/c_2)^{1/2k} \leq \dots$$

Consequently if $\sum_{k \geq 1} (c_{2k+2}/c_2)^{1/2k} = \infty$ then by Carleman's criterion coefficients c_i identify measure dK and consequently Lévy process itself.

$$ii) c_4/c_2 - (c_3/c_2)^2 \geq 0 \text{ since } c_4/c_2 - (c_3/c_2)^2 \text{ is the variance of measure } \frac{1}{c_2} dK(u),$$

iii) if $c_{2k} = 0$ for some $k \geq 2$ then dK must be degenerated and concentrated at 0 so we deal with the Gaussian case since

$$\exp(c_1xt + tc_2x^2/2) = \int \exp(xy) \frac{1}{\sqrt{2\pi c_2 t}} \exp\left(-\frac{(y - xc_1t)^2}{2c_2t}\right) dy.$$

Remark 2. Following interpretation of coefficients c_i given above we can refer to the martingale characterizations given by Wesolowski in [26]. One of them is obviously wrong. Namely the characterization of the Poisson process by the form of first three polynomial martingales is not true. This is so since from the martingale conditions considered by Wesolowski in Theorem 1. it follows that $c_1 = c_2 = c_3 = 1$. As the above Remark shows it is not enough to impose that all $c_i = 1$ for $i \geq 4$ which would lead to the Poisson process with parameter 1.

On the other hand the second martingale characterization of the Wiener process by the first four polynomial martingales given by Theorem 3. is true since the form of these martingales impose that $c_3 = c_4 = 0$. As it can be seen from Remark 1 it is enough to deduce that then all $c_i = 0$ for $i \geq 4$.

Remark 3. In [27](2.1) Yablonski defined family of polynomials $P_n(x_1, \dots, x_n)$ of the increasing numbers of variables by the expansion

$$(2.5) \quad \exp\left(\sum_{k \geq 1} \frac{(-1)^{k-1} x_k}{k} z^k\right) = \sum_{n \geq 0} z^n P_n(x_1, \dots, x_n).$$

He proved validity of the above expansion for $|z| < 1/\limsup_{k \rightarrow \infty} |x_k|^{1/k}$ and also gave some properties of these polynomials. Comparing (2.5) with Proposition 1,ii) we see that

$$(2.6) \quad m_n(t) = n! P_n(c_1 t, -c_2 t, c_3 t/2, \dots, (-1)^{n-1} t c_n / (n-1)!),$$

where P_n is the above mentioned Yablonski's polynomial. Moreover Yablonski condition for the existence of expansion (2.5) can be expressed in terms of variables c_i in the following form : $\limsup_{k \rightarrow \infty} \frac{|x_k|^{1/k}}{k} < \infty$ or equivalently $\frac{1}{|x_k|^{1/k}} \geq \frac{D}{k}$ for some D and sufficiently large k which is nothing else as Carleman's condition for identifiability of the Kolmogorov's measure by moments and consequently of the marginal distribution of process since Kolmogorov's measure identifies this distribution and parameters c_i are moments of dK .

Following formulae ([27], (2.2)–(2.4)) and using our notation given by (2.6) we have the following properties of moments $m_n(t)$:

$$(2.7) \quad m_{n+1}(t) = t \sum_{j=0}^n \binom{n}{j} c_{j+1} m_{n-j}(t),$$

$$(2.8) \quad \frac{\partial m_n(t)}{\partial c_l} = \begin{cases} 0 & \text{if } l > n \\ n t m_{n-l}(t) & \text{if } l \leq n \end{cases},$$

$$(2.9) \quad m_n(t; \mathbf{c} + \mathbf{d}) = \sum_{k=0}^n \binom{n}{k} m_k(t; \mathbf{c}) m_{n-k}(t; \mathbf{d}),$$

$$(2.10) \quad m_n(t; (c_1 \alpha, c_2 \alpha^2, \dots)) = \alpha^n m_n(t; (c_1, c_2, \dots)).$$

where we denoted $m_n(t; \mathbf{c})$ n -th moment of the Lévy process with parameters $\mathbf{c} = (c_1, c_2, \dots)$.

Finally let us remark that as shown in [15] Yablonski's polynomials P_n are closely related to Kailath–Segall polynomials formulae (see [14]) that are used to study path properties of Lévy processes. Hence our results give new interpretation of these polynomials.

Remark 4. Since matrices \mathcal{M}_n are lower triangular we see that their (i, j) entries do not depend on n . Consequently from formula (2.4) it follows that : $(M_n(X_t, t), \mathcal{F}_{\leq t})$, where

$$(2.11) \quad M_n(X_t, t) = \sum_{j=0}^n \binom{n}{j} m_{n-j}(-t) X_t^j$$

are martingales for $n = 1, 2, \dots$ and $EM_n(X_t, t) = EM_n(X_0, 0) = 0$.

By $M_n(x, t)$ we will denote polynomial such that $M_n(X_t, t)$ is defined by (2.11).

Using this formula and (2.7) we have the following set of useful relationships:

Lemma 1. i)

$$M_1(x, t) M_n(x, t) = M_{n+1}(x, t) + t \sum_{k=1}^n \binom{n}{k} c_{k+1} M_{n-k}(x, t).$$

Thus in particular $EM_1(X_t, t)M_n(X_t, t) = tc_{n+1}$.

ii)

$$M_2(x, t)M_n(x, t) = M_{n+2}(x, t) + 2nc_2tM_n(x, t) + t \sum_{k=2}^{n+1} \binom{n}{k-1} \\ + 2 \binom{n}{k} c_{k+1} M_{n-k+1}(x, t) + t^2 \sum_{l=2}^n \binom{n}{l} M_{n-l}(x, t) \sum_{k=1}^{l-1} \binom{l}{k} c_{k+1} c_{l-k+1}.$$

In particular $EM_2(X_t, t)M_n(X_t, t) = tc_{n+2} + t^2 \sum_{k=1}^{n-1} \binom{n}{k} c_{k+1} c_{n+1-k}$.

iii) $\forall \alpha \neq 0 : M_n(x/\alpha, t; (c_1, \dots, c_n)) = \alpha^{-n} M_n(x, t; (\alpha c_1, \alpha^2 c_2, \dots, \alpha^n c_n))$, where we denoted $M_n(x, t; (c_1, \dots, c_n)) = \sum_{j=0}^n \binom{n}{j} m_{n-j}(-t, (c_1, \dots, c_{n-j})) x^j$.

Proof. Uninteresting proof is shifted to Section 4. \square

As a immediate corollary we get the following nice property of Lévy processes

Theorem 1. Let $\mathbf{X}(\{c_1, c_2, \dots\})$ be some Lévy process defined on $(0, \infty)$ and polynomial martingales defined by (2.4).

Then $(M_1(X_t, t)/t, \mathcal{F}_{\leq t})$ is a reversed martingale. Moreover M_1 is also a harness: i.e.

$$E(M_1(X_t, t) | \mathcal{F}_{s,u}) = \frac{u-t}{u-s} M_1(X_s, s) + \frac{t-s}{u-s} M_1(X_u, u),$$

where $s < t < u$, and $\mathcal{F}_{s,u} = \sigma(X_v; v \in (0, s] \cup [u, \infty))$.

Proof. Simple proof strongly basing on Lemma 1,i) is shifted to Section 4. \square

We will need also some simple properties of moments $m_n(t)$ and martingales $M_n(t)$.

Proposition 2. Let $Q(s; x) = \exp(sf(x))$ where $f(x) = \sum_{k \geq 1} c_k \frac{x^k}{k!}$, be the characteristic function of moments $m_n(t)$. Then

$$i) \sum_{j=0}^n \binom{n}{j} m_{n-j}(-s) m_{j+i}(s) = \exp(-sf(u)) \frac{\partial^i}{\partial u^i} \exp(sf(u)) \Big|_{u=0} \\ ii) EM_n(X_t, t) M_k(M_t, t) = \frac{\partial^n \partial^k}{\partial u^n \partial v^k} \exp(t(f(u+v) - f(u) - f(v))) \Big|_{u=v=0} \\ = \frac{\partial^n}{\partial u^n} \frac{\partial^k}{\partial v^k} \exp(t(f(u+v) - f(u) - f(v))) \Big|_{v=0} \Big|_{u=0}.$$

Proof. Purely technical proof is sifted to Section 4. \square

Corollary 1. $EM_n(X_t, t) M_k(M_t, t)$ is a polynomial in t of order $\min(n, k)$ with coefficient by t^j equal to $\frac{d^{n+k-j}}{dx^{n+k-j}} (f'(x))^j \Big|_{x=0}$ for $j = 0, \dots, \min(n, k)$. More preciselly $EM_n(X_t, t) M_k(X_t, t) = \sum_{j=1}^{\min(n, k)} d_j^{(k, n)} t^j$, with

$$d_j^{(k, n)} = \begin{cases} \frac{d^{2k-j}}{dx^{2k-j}} (f'(x))^j \Big|_{x=0} & \text{if } n = k \\ \frac{d^{n+k-j}}{dx^{n+k-j}} (f'(x))^j \Big|_{x=0} & \text{if } n \neq k \end{cases}.$$

In particular coefficient by t is equal to c_{n+k} , by t^2 $\frac{d^{n+k-2}}{dx^{n+k-2}} (f'(x))^2 \Big|_{x=0}$ and by $t^{\min(n, k)}$ is equal to $\frac{d^{\max(n, k)}}{dx^{\max(n, k)}} (f'(x))^{\min(n, k)} \Big|_{x=0}$. If $n = k$ coefficient by t^k is equal to $k! c_2^k > 0$. In all these expressions we have to set $c_1 = 0$.

Proof. Firstly we observe that n -th derivative of $\exp(tf(x))$ with respect to x is of the form $(tf^{(n)}(x) + \dots + t^n(f'(x))^n)\exp(tf(x))$. The independence of c_1 follows the fact that $\exp(t(f(u+v) - f(u) - f(v)))$ does not depend on c_1 . \square

Since coefficients c_i , $i \geq 1$ determine Lévy process with finite all moments completely we will use notation $\mathbf{X}(\{c_i\})$, $\mathbf{X}(\mathbf{c})$, or finally $\mathbf{X}(\{c_1, c_2, \dots\})$ to denote Lévy process with parameters $\{c_1, c_2, \dots\}$.

Our main concern in this paper is to select those Lévy processes with all moments existing that have also polynomial reversed martingales and orthogonal martingales (that necessarily are also reversed martingales as remarked in [21], Corollary 5).

This problem is too complex to be solved in full generality in a short paper. Instead we will solve it only partially. Namely we will select those polynomial martingales $M_n(x, t)$ that multiplied by some deterministic function $\mu_n(t)$ constitute a reversed martingale.

Our main result states that for $n \geq 3$ within the class of Lévy processes with all moments only ones with all parameters c_i for $i \geq 3$ equal to zero have this property. Moreover then $M_n(x, t)$ are orthogonal martingales.

For $n = 2$ we show that apart Wiener process (i.e. apart from the case $c_i = 0$ for $i \geq 3$) there exist another Lévy process that has the property that there exists a function $\mu(t)$ such that $\mu(t)M_2(X_t, t)$ is a reversed martingale.

Both these theorems will be based on the some simple observations that we will collect in the following lemma.

Lemma 2. *Let $\mathbf{X}(\{c_1, c_2, \dots\})$ be Lévy process defined on $(0, \infty)$ and polynomial martingales defined by (2.4). Suppose $\mu(t)M_k(t)$ is the reversed martingale, then*

$$(2.12) \quad \mu(s)EM_l(X_s, s)M_k(X_s, s) = \mu(t)EM_l(X_t, t)M_k(X_t, t),$$

where $\mu(t) = 1/EM_k(X_t, t)M_k(X_t, t)$, and $EM_l(X_t, t)M_k(X_t, t) = \sum_{j=1}^{\min(k,l)} d_j^{(k,l)} t^j$, with

$$d_j^{(k,l)} = \begin{cases} 0 & \text{if } l < k \\ \left. \frac{d^{2k-j}}{dx^{2k-j}} (f'(x))^j \right|_{x=0} & \text{if } l = k \\ \left. \frac{d^{l+k-j}}{dx^{l+k-j}} (f'(x))^j \right|_{x=0} & \text{if } l > k \end{cases}.$$

ii) $c_j = 0$, $j = \max(3, k-1), \dots, 2k-1$ for $k \geq 2$.

Proof. Is shifted to Section 4. \square

Remark 5. *Let us notice that polynomials $p_k(t) = EM_k(X_t, t)M_k(X_t, t)$ are in fact so called 'angular brackets' of polynomial martingales $M_k(X_t, t)$. We know that they are non-decreasing functions of t .*

As an immediate corollary of Lemma 2,ii) and Remark 1,iii) we have the following result.

Theorem 2. *For $k \geq 3$ there does not exist function $\mu(t)$ such that $\mu(t)M_k(t)$ is a reversed martingale unless $c_i = 0$ for $i \geq 3$. Also this condition is necessary and sufficient for the polynomial martingales $M_n(t)$ to constitute a family of orthogonal polynomials.*

Proof. By Lemma 2 we know that parameters $c_3, c_4, \dots, c_{2k-1}$ are equal to zero. In particular we have $c_4 = 0$ which leads by Remark 1,iii) to the conclusion that $c_i = 0$ for $i \geq 3$.

Notice that to have orthogonal polynomial martingales we have to have $EM_l(X_s, s)M_k(X_s, s) = 0$ for $k \neq l$. The presented above consideration show that it is possible only iff $c_i = 0$ for $i \geq 3$. \square

Thus it remains to consider the case $k = 2$.

Theorem 3. $(\mu(t)M_2(X_t, t), \mathcal{F}_{\leq t})$ is a reversed martingale with $\mu(t) = 1/(2c_2^2t + c_4)$ iff either $c_4 = 0$ and the Lévy process is Gaussian or $c_4 > 0$ and

$$(2.13) \quad \exp(tf(x)) = e^{c_1tx} (\cos(x\sqrt{\frac{c_4}{2c_2}}))^{-2tc_2^2/c_4},$$

for $|x| < \frac{\pi}{2}\sqrt{\frac{2c_2}{c_4}}$. In particular distribution of X_t for $t = \frac{c_4}{2c_2^2}$ has density $h(y)$ equal to

$$(2.14) \quad h(y) = \frac{\sqrt{c_4}}{\sqrt{8c_2} \cosh(\frac{\pi y \sqrt{2c_2}}{2\sqrt{c_4}})}; \quad y \in \mathbb{R}.$$

and is identifiable by moments.

As a corollary we can now refer to the third martingale characterization of the Wiener process done by Wołowski in [25]. It states that if a square integrable process $\mathbf{X} = (X_t)_{t \geq 0}$ has the property that $(X_t, \mathcal{F}_{\leq t})$ and $(X_t^2 - t, \mathcal{F}_{\leq t})$ are martingales and $(X_t/t, \mathcal{F}_{\geq t})$ and $((X_t^2 - t)/t^2, \mathcal{F}_{\geq t})$ are reversed martingales then the process is a Wiener process. It was shown in [23] that this is not true characterization. Namely a counterexample with dependent increments was shown.

If we however confine ourselves to class of Lévy processes having all moments then this characterization is true. Since as shown above for our class of Lévy processes with $c_1 = 0, c_2 = 1$, $(X_t, \mathcal{F}_{\leq t})$ and $(X_t^2 - t, \mathcal{F}_{\leq t})$ are martingales and $(X_t/t, \mathcal{F}_{\geq t})$ is reversed martingale only condition that $((X_t^2 - t)/t^2, \mathcal{F}_{\geq t})$ is a reversed martingale matters. Comparing this requirement with Theorem 3 we see that we must have $c_4 = 0$ to fulfill the requirement. But $c_4 = 0$ leads to requirement that $c_i = 0$, for all $i \geq 3$.

3. OPEN PROBLEMS

Notice that in fact one should seek reversed polynomial martingales $R_n(X_t, t)$ of the form

$$R_n(X_t, t) = \sum_{k=1}^n \mu_k(t) M_k(X_t, t),$$

where $M_k(X_t, t)$ are polynomial martingales that were considered in the previous sections and $\mu_k(t)$ are some deterministic polynomials of t . Then requirement that $(R_n(X_t, t), \mathcal{F}_{\geq t})$ is the reversed martingale leads to the condition that for all $s < t$, $l \geq 1$ we have

$$(3.1) \quad \sum_{k=1}^n \mu_k(s) EM_k(X_s, s) M_l(X_s, s) = \sum_{k=1}^n \mu_k(t) EM_k(X_t, t) M_l(X_t, t).$$

Since $M_i(X_t, t)$ for $i = 1, \dots, n$ is martingale.

For $n = 2$ we get

$$(3.2) \quad \mu_2(t) = \frac{c_2 - \beta c_3}{t(2c_2^3t + c_2c_4 - c_3^2)},$$

$$(3.3) \quad \mu_1(t) = \frac{\beta(2c_2^2t + c_4) + c_3}{t(2c_2^3t + c_2c_4 - c_3^2)},$$

where parameter β is defined by the condition $\mu(t)tc_3 + \nu(t)tc_2 = \beta$ which comes from (3.1) taken for $n = 2$ and $l = 1$.

Notice that if we assume $c_3 = 0$ then

$$\mu_2(t) = \frac{1}{t(2c_2^2t + c_4)}, \quad \mu_1(t) = \frac{\beta}{c_2t}.$$

But since we know that $M_1(X_t, t)/t$ is a reversed martingale by Theorem 1, thus the requirement that $\mu_2(t)M_2(X_t, t) + \beta M_1(X_t, t)/(c_2t)$ is a reversed martingale leads to the requirement that $\mu_2(t)M_2(X_t, t)$ is. Thus we deal with the case already considered in Theorem 3. Hence we have an open problem

Problem 1. Find all Lévy processes (i.e. all coefficients c_i , $i \geq 5$) such that (3.1) is satisfied for $n = 2$, $l \geq 3$ with (3.2) and (3.3). Little reflection leads to the conclusion that then coefficients c_i must satisfy recursion:

$$c_{l+2} = -\frac{c_3}{c_2}c_{l+1} + \frac{(c_2c_4 - c_3^2)}{2c_2^3} \sum_{k=1}^{l-1} \binom{l}{k} c_{k+1}c_{l+1-k}.$$

Of course one can pose also a generalization of the previous open problem.

Problem 2. Find all Lévy processes (i.e. all coefficients c_i , $i \geq 5$) such that (3.1) is satisfied for $n \geq 3$, $l \geq n + 1$.

Here Lemma 2,i) might be of help.

Similarly one can pose the following problem concerning so called quadratic harnesses among Lévy process the problem inclusively studied recently by Bryc, Wesolowski and Matysiak (see [4],[3],[6],[7],[8],[10],[5]).

Problem 3. Find all Lévy process (i.e. coefficients c_i , $i \geq 3$) such that $M_2(X_t, t)$ is a quadratic harness i.e.

$$\begin{aligned} E(M_2(X_t, t) | \mathcal{F}_{s,u}) &= AM_2(X_s, s) + BM_1(X_s, s)M_1(X_u, u) \\ &+ CM_2(X_u, u) + DM_1(X_s, s) + EM_1(X_u, u) - c_2s^2, \end{aligned}$$

where $0 < s < t < u$, A, B, C, D, E are some functions of s, t, u only. Note that A, B, C, D, E can be relatively easy found by solving system of 5 linear equations obtained by multiplying the above equality by $M_2(X_s, s)$, $M_2(X_u, u)$, $M_1(X_s, s)M_1(X_u, u)$, $M_1(X_s, s)$ and $M_1(X_u, u)$ and calculating expectation of both sides and utilizing the fact that $M_i(X_t, t)$, $i = 1, 2$ are martingales. (as done in [21]). Having A, B, C, D, E we multiply both sides of this equality by $M_l(X_s, s)M_k(X_u, u)$ and calculating expectation. On the way we use Lemma 1i)-ii) and Lemma 2,i). In this way we get system of recursions to be satisfied by coefficients c_i , $i \geq 4$.

4. PROOFS

Proof of Proposition 1. i) We have $m_n(s+t) = EX_{t+s}^n = E(X_{t+s} - X_s + X_s)^n = \sum_{j=0}^n \binom{n}{j} m_j(t) m_{n-j}(s)$ since $E(X_{t+s} - X_s)^n = m_n(t)$ for Lévy processes.

ii) Let us define $Q(t; x) = \sum_{j \geq 0} m_n(t) x^j / j!$. Following i) we get

$$Q(t+s; x) = Q(t; x)Q(s; x).$$

Since for fixed x function Q is continuous in the first argument by assumption we are dealing with multiplicative Cauchy equation. Hence $Q(t; x) = \exp(tf(x))$. Since $Q(t; x)$ is analytic with respect to x and also $Q(t; 0) = 1$ we expand function f in a power series of the form $f(x) = \sum_{k \geq 1} c_k x^k / k!$. Following definition of the function Q we get further statements of ii).

iii) We have by direct calculation: $m_1(t) = EX_t = \frac{\partial}{\partial x} \exp(tf(x))|_{x=0} = c_1 t$ and $m_2(t) = EX_t^2 = \frac{\partial^2}{\partial x^2} \exp(tf(x))|_{x=0} = c_1^2 t + c_2 t$. Now let us consider sequence $\hat{m}_n(t)$. We have $\hat{m}_n(t) = \sum_{i=0}^n \binom{n}{i} m_{n-i}(t) (-1)^i (c_1 t)^i$ and also $\sum_{i \geq 0} (-1)^i (c_1 t)^i \frac{x^i}{i!} = \exp(-c_1 t x)$. Hence $\sum_{j \geq 0} \hat{m}_n(t) \frac{x^n}{n!} = \exp(tf(x) - c_1 t x)$.

iv) First of all notice that following definition of function Q we have $m'_n(t) = \frac{\partial^n \partial Q(t; x)}{\partial x^n \partial t} \Big|_{x=0} = \frac{\partial^n}{\partial x^n} (f(x) \exp(tf(x))) \Big|_{x=0}$. Now we apply Leibnitz formula for n -th derivative of the product of two differentiable functions. On the way have to remember that $\frac{d^n}{dx^n} f(x) \Big|_{x=0} = c_n$.

v) At this point we have to refer to [21]. There one of the basic notions was the family of the so called structural matrices $\{V_n(t)\}_{n \geq 0}$ of the process with polynomial conditional moment. Matrix $V_n(t)$ was defined as any $(n+1) \times (n+1)$ matrix V_n satisfying the relationship

$$(4.1) \quad \hat{A}_n(s, u) = V_n^{-1}(s) V_n(u),$$

where $\hat{A}_n(s, t) = [\hat{\gamma}_{i,j}(s, t)]_{i,j=0,2,\dots,n}$ $n \geq 0$ and functions $\gamma_{n,k}(s, t)$ are defined by the relationship: $E(X_t^n | \mathcal{F}_{\leq s}) = \sum_{k=0}^n \hat{\gamma}_{n,k}(s, t) X_s^k$ for all $s < t$ and all $n \geq 1$. Little reflection lead to the conclusion that for Lévy processes coefficients $\hat{\gamma}_{n,k}(s, t)$ have clear probabilistic interpretation. Namely $\gamma_{n,k}(s, t) = \binom{n}{k} E(X_t - X_s)^{n-k} = \binom{n}{k} m_{n-k}(t-s)$. Besides it was shown in [21] (Corollary 4) that (i, j) -th entry of matrix V_n for process with independent increments is of the form:

$$v_{i,j}(t) = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \\ \binom{i}{j} g_{i-j}(t) & \text{if } i > j \end{cases}.$$

Notice that equation (4.1) can be read as $V_n(t) = \hat{A}_n(s, t) V_n(s)$ leading to the conclusion that we must have:

$$\binom{i}{j} g_{i-j}(t) = \sum_{k=j}^i \binom{i}{k} m_k(t-s) \binom{k}{j} g_{k-j}(s)$$

for all $t > s$ and $i \geq j$. Canceling out by $\binom{i}{j}$ and changing index of summation we get:

$$g_{i-j}(t) = \sum_{l=0}^{i-j} \binom{i-j}{l} m_{i-j-l}(t-s) g_l(s).$$

Comparing this with (2.1) we deduce that as matrix $V_n(t)$ we may take matrix with (i, j) -th entry equal to $\binom{i}{j} m_{i-j}(t)$ for all $i, j = 0, \dots, n$ and $t \geq 0$. Now let us calculate matrix $V'_n(t)$. By iii) we see that its (i, j) -entry is equal to $\binom{i}{j} m'_{i-j}(t) = \binom{i}{j} \sum_{k=1}^{i-j} \binom{i-j}{k} c_k m_{i-j-k}(t)$. On the other hand (i, j) -th entry of the product $C_n V_n(t)$ is equal to $\sum_{k=j}^i \binom{i}{k} c_{i-k} \binom{k}{j} m_{k-j}(t)$. After changing index of summation from k to $t = j - k$ we get further $\binom{i}{j} \sum_{t=0}^{i-j} \binom{i-j}{t} c_{i-j-t} m_t$ but this is equal to previously calculated $\binom{i}{j} m'_{i-j}(t)$ because of the symmetry properties of the binomial coefficient. We have thus shown that

$$V'_n(t) = C_n V_n(t),$$

from which immediately follows (2.3). The fact that polynomials defined by \mathcal{M}_n are martingales follows Corollary 4. Besides as it follows from (2.3) $V_n^{-1}(t) = V_n(-t)$ for Lévy processes. \square

Proof of Lemma 1. i) We have

$$\begin{aligned} xM_n(x, t) &= \sum_{j=0}^n \binom{n}{j} m_{n-j}(-t) x^{j+1} = \sum_{j=1}^{n+1} \binom{n}{j-1} m_{n+1-j}(-t) x^j \\ &= M_{n+1}(x, t) - m_{n+1}(-t) + \sum_{j=1}^n \left(\binom{n}{j-1} - \binom{n+1}{j} \right) m_{n+1-j} x^j \\ &= M_{n+1}(x, t) - \sum_{j=0}^n \binom{n}{j} x^j m_{n-j+1}(-t) = M_{n+1}(x, t) - \sum_{j=0}^n \binom{n}{j} x^{n-j} m_{j+1}(-t) \\ &= M_{n+1}(x, t) + t \sum_{j=0}^n \binom{n}{j} x^{n-j} \sum_{k=0}^j \binom{j}{k} c_{k+1} m_{j-k}(-t) \\ &= M_{n+1}(x, t) + t \sum_{k=0}^n \binom{n}{k} c_{k+1} \sum_{j=k}^n \binom{n-k}{j-k} x^{n-j} m_{j-k}(-t) \\ &= M_{n+1}(x, t) + t \sum_{k=0}^n \binom{n}{k} c_{k+1} \sum_{l=0}^{n-k} \binom{n-k}{l} x^{n-k-l} m_l(-t). \end{aligned}$$

Now recall that $M_1(x, t) = x - c_1 t$, hence subtracting from both sides $c_1 t M_n(x, t)$ get the assertion. Besides we have $EM_n(X_t, t) = 0$ for $n \geq 1$.

ii) Recall that $M_2(x, t) = M_1(x, t)^2 - c_2 t$, hence let us calculate first $M_1(x, t)^2 M_n(x, t)$

$$\begin{aligned}
M_1(x, t)^2 M_n(x, t) &= M_1(x, t) M_{n+1}(x, t) + t \sum_{k=1}^n \binom{n}{k} c_{k+1} M_{n-k}(x, t) M_1(x, t) \\
&= M_{n+2}(x, t) + t \sum_{k=1}^{n+1} \binom{n+1}{k} c_{k+1} M_{n+1-k}(x, t) + t \sum_{k=1}^n \binom{n}{k} c_{k+1} M_{n-k+1}(x, t) + \\
&\quad t^2 \sum_{k=1}^{n-1} \binom{n}{k} c_{k+1} \sum_{j=1}^{n-k} \binom{n-k}{j} c_{j+1} M_{n-k-j}(x, t) \\
&= M_{n+2}(x, t) + t \sum_{k=1}^{n+1} \left(\binom{n}{k-1} + 2 \binom{n}{k} \right) c_{k+1} M_{n-k+1}(x, t) + \\
&\quad t^2 \sum_{l=2}^n \binom{n}{l} M_{n-l}(x, t) \sum_{k=1}^{l-1} \binom{l}{k} c_{k+1} c_{l-k+1}
\end{aligned}$$

Since $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

iii) We have $M_n(x/\alpha, t; (c_1, \dots, c_n)) = \sum_{j=0}^n \binom{n}{j} m_{n-j}(-t; (c_1, \dots, c_{n-j})) \alpha^{-j} x^j = \alpha^{-n} \sum_{j=0}^n \binom{n}{j} \alpha^{n-j} m_{n-j}(-t; (c_1, \dots, c_{n-j})) x^j = \alpha^{-n} M_n(x, t; (\alpha c_1, \dots, \alpha^n c_n))$ by (2.10). \square

Proof of Theorem 1. To see that $M_1(X_t, t)/t$ is a reversed martingale we have to show that for all $s < t$ and l we have :

$$\frac{1}{s} E M_1(X_s, s) M_l(X_s, s) = \frac{1}{t} E M_1(X_t, t) M_l(X_t, t).$$

By Lemma 1,i) we see that this is satisfied. To prove the 'harness' part we have to show that for all $s < t < u$ and all k and $l \geq 1$ we have

$$\begin{aligned}
E M_k(X_s, s) M_1(X_t, t) M_l(X_u, u) &= \frac{u-t}{u-s} E M_k(X_s, s) M_1(X_s, s) M_l(X_u, u) \\
&\quad + \frac{t-s}{u-s} E M_k(X_s, s) M_1(X_u, u) M_l(X_u, u).
\end{aligned}$$

By Lemma 1,i) and the fact that polynomials $M_i(X_t, t)$ are martingales we see that:

$$\begin{aligned}
E M_k(X_s, s) M_1(X_t, t) M_l(X_u, u) &= E M_k(X_s, s) M_{l+1}(X_s, s) \\
&\quad + t \sum_{j=1}^l \binom{l}{j} c_{j+1} E M_k(X_s, s) M_{l-j}(X_s, s), \\
E M_k(X_s, s) M_1(X_s, s) M_l(X_u, u) &= E M_k(X_s, s) M_{l+1}(X_s, s) \\
&\quad + s \sum_{j=1}^l \binom{l}{j} c_{j+1} E M_k(X_s, s) M_{l-j}(X_s, s), \\
E M_k(X_s, s) M_1(X_u, u) M_l(X_u, u) &= E M_k(X_s, s) M_{l+1}(X_s, s) \\
&\quad + u \sum_{j=1}^l \binom{l}{j} c_{j+1} E M_k(X_s, s) M_{l-j}(X_s, s).
\end{aligned}$$

Since $\frac{u-t}{u-s} + \frac{t-s}{u-s} = 1$ and $\frac{u-t}{u-s} s + \frac{t-s}{u-s} u = t$ we get our assertion. \square

Proof of Proposition 2. i) We have:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{u^n}{n!} \sum_{j=0}^n \binom{n}{j} m_{n-j}(-s) m_{j+i}(s) = \\ \exp(-sf(x)) \sum_{j=0}^{\infty} \frac{u^j}{j!} m_{j+i}(s) = \exp(-sf(u)) \frac{\partial^i}{\partial u^i} \exp(sf(u)). \end{aligned}$$

ii) Recall that

$$\exp(tf(x)) = E \sum_{k \geq 0} \frac{x^k}{k!} X_t^k = E \exp(xX_t).$$

Hence

$$\begin{aligned} \sum_{j \geq 0} \sum_{k \geq 0} \frac{u^j v^k}{j! k!} M_j(t) M_k(t) &= E \sum_{j \geq 0} \sum_{k \geq 0} \frac{u^j v^k}{j! k!} \sum_{l=0}^j \binom{j}{l} m_{j-l}(-t) X_t^l \sum_{s=0}^k \binom{k}{s} m_{k-s}(-t) X_t^s = \\ &= E \sum_{j \geq 0} \frac{u^j}{j!} \sum_{l=0}^j \binom{j}{l} m_{j-l}(-t) X_t^l \sum_{s \geq 0} \frac{v^s}{s!} X_t^s \sum_{k \geq s} \frac{v^{k-s}}{(k-s)!} m_{k-s}(-t) = \\ &= \exp(-tf(v)) E \exp(vX_t) \sum_{j \geq 0} \frac{u^j}{j!} \sum_{l=0}^j \binom{j}{l} m_{j-l}(-t) X_t^l = \\ &= \exp(-tf(v) - tf(u)) E \exp((v+u)X_t) = \exp(-tf(v) - tf(u)) \exp(tf(u+v)). \end{aligned}$$

To see the authenticity of the second equality notice that

$\frac{\partial^k}{\partial v^k} \exp(t(f(u+v) - f(u) - f(v))) \Big|_{u=0} = 0$ for $k \geq 1$. Further notice that $\frac{\partial^k}{\partial v^k} \exp(t(f(u+v) - f(u) - f(v)))$ is a product of two expressions : first being a polynomial in t of order k with coefficients being some differential expressions of $f(u+v) - f(v)$ and the second $\exp(t(f(u+v) - f(u) - f(v)))$. Consequently upon applying Leibnitz rule to this product and setting $u = v = 0$ we see that only the first expression matters. The assertion follows the fact that $\frac{\partial^n}{\partial u^n} (f^{(j)}(u+v) - f^{(j)}(v)) \Big|_{u=v=0} = \frac{\partial^n}{\partial u^n} (f^{(j)}(u+v) - f^{(j)}(v)) \Big|_{v=0} \Big|_{u=0}$ for $j = 1, \dots, k$. \square

Proof of Lemma 2. First of all notice that if $\mu(t)M_k(t)$ is a reversed martingale then $E(\mu(s)M_k(X_s, s) | \mathcal{F}_{\geq t}) = \mu(t)M_k(X_t, t)$ a.s., hence multiplying both sides by $M_l(X_t, t)$ and taking expectation of both sides we get $\mu(s)EM_k(X_s, s)M_l(X_t, t) = \mu(t)EM_k(X_t, t)M_l(X_t, t)$. Finally we use the fact that M_l is a martingale. Thus we get (2.12). By Corollary 1 we know that $EM_k(X_t, t)M_l(X_t, t)$ is a polynomial of order $\min(k, l)$ in t . Moreover if $l = k$ coefficient by t^k is equal to $k!c_2^k > 0$. Secondly notice that quantity $\mu(t)EM_k(X_t, t)M_l(X_t, t)$ has to be independent on t , thus since for $l = k$ $EM_k(X_t, t)M_l(X_t, t)$ is a polynomial in t of exactly k -th order we deduce that $\mu(t)$ must be proportional to the inverse of $EM_k(X_t, t)M_k(X_t, t)$.

i) By Corollary 1 we know that for $l < k$ $EM_k(X_t, t)M_l(X_t, t)$ is a polynomial in t of order l , so if $\mu(t)EM_k(X_t, t)M_l(X_t, t)$ is to be independent of t $EM_k(X_t, t)M_l(X_t, t)$ must be zero polynomial.

ii) The fact that $c_{k+l} = 0$, $l = 1, \dots, k-1$ follows formula $d_1^{(k,l)} = c_{k+l}$ and the fact that $EM_k(X_t, t)M_l(X_t, t)$ for $l < k$ must be zero polynomial in particular its coefficients by t (which are equal to c_{k+l}) must be equal to zero. In this way we

get the case $k = 2$. Let us now consider coefficient in $EM_k(X_t, t)M_l(X_t, t)$ by t^2 . It is equal to $\sum_{j=1}^{l+k-3} \binom{l+k-2}{j} c_{j+1} c_{l+k-j-1}$ as indicated by Remark ???. Let us now take into account the fact that c_{k+1}, \dots, c_{2k-1} are equal to zero. It means that in fact we have to have: $\sum_{j=l-1}^{k-1} \binom{l+k-2}{j} c_{j+1} c_{l+k-j-1} = 0$. Now we change index of summation to $s = j - l + 1$ and get that for all $l = 2, \dots, k - 1$ we have to have $\sum_{s=0}^{k-l} \binom{k+l-2}{s+l-1} c_{s+l} c_{k-s} = 0$. Let us consider $l = k - 1$ and $k - 2$. From the first equality we deduce that $c_k c_{k-1} = 0$ and from the second that $((\binom{2k-4}{k-3} + \binom{2k-4}{k-1})) c_{k-2} c_k + \binom{2k-4}{k-2} c_{k-1}^2 = 0$. Now if $k = 3$ and $c_2 > 0$ we deduce that $c_3 = 0$ when $k = 3$. Thus let us take $k \geq 4$. By multiplying both sides of the last equality by c_{k-1} we deduce that since $c_{k-1} c_k = 0$ that $c_{k-1} = 0$, or equivalently that $c_{k-2} c_k = 0$. Let us consider now $l = k - 3$. We get $((\binom{2k-5}{k-4} + \binom{2k-5}{k-1})) c_{k-3} c_k + ((\binom{2k-5}{k-3} + \binom{2k-5}{k-2})) c_{k-2} c_{k-1} = 0$. Hence $c_k c_{k-3} = 0$ and so on. But after $k - 2$ such steps we will get $c_2 c_k = 0$. But $c_2 > 0$. So we deduce that $c_k = 0$. \square

Proof of the Theorem 3. We will extensively use assertions of Lemma 1. By Lemma 2 we know that

$$\mu(t) = 1/EM_2^2(X_t, t) = \frac{1}{t(2c_2^2 t + c_4)},$$

and that if $\mu(t) M_2(X_t, t)$ is to be a martingale then we have to have equality for all $l \geq 1$.

$$\mu(t) EM_2(X_t, t) M_l(X_t, t) = \mu(s) EM_2(X_s, s) M_l(X_s, s).$$

By Lemma 1,ii) we see that we have to have for $0 < s < t$:

$$\begin{aligned} & st(2c_2^2 s + c_4)(c_{l+2} + t \sum_{k=1}^{l-1} \binom{l}{k} c_{k+1} c_{l+1-k}) \\ &= st(2c_2^2 t + c_4)(c_{l+2} + s \sum_{k=1}^{l-1} \binom{l}{k} c_{k+1} c_{l+1-k}), \end{aligned}$$

which leads to the condition: $2c_2^2 c_{l+2} - c_4 \sum_{k=1}^{l-1} \binom{l}{k} c_{k+1} c_{l+1-k} = 0$. Since $c_2 > 0$ we can rewrite this condition in the following way:

$$(4.2) \quad c_{l+2} = \frac{c_4}{2c_2^2} \sum_{j=2}^l \binom{l}{j-1} c_j c_{l+2-j}.$$

for all $l \geq 1$. But for $l = 1$ we have $c_3 = 0$ and (4.2) is satisfied. Besides notice that if l is odd the right hand side of (4.2) depends on c_k with odd indices $k \leq l$. Consequently we deduce that c_{2k+1} must be zero since $c_3 = 0$. Hence let us consider only parameters c_k with even indices. We know that if $c_4 = 0$ then we have to have $c_k = 0$ for all $k \geq 4$ so (4.2) is satisfied. So let us assume that $c_4 > 0$. Let us rewrite this condition in the following way:

$$(4.3) \quad c_{2k+2} = \frac{c_4}{2c_2^2} \sum_{j=0}^{k-1} \binom{2k}{2j+1} c_{2j+2} c_{2k-2j},$$

Since all c_j with odd indices are equal to zero we have $\sum_{j=0}^{k-1} \binom{2k}{2j+1} c_{2j+2} c_{2k-2j}$. Consequently coefficients c_{2k} must satisfy recursion: for $k \geq 1$.

One can easily notice that $c_6 = \frac{c_4}{2c_2^2} 8c_2c_4 = 4c_2(\frac{c_4}{c_2})^2$ and consequently that c_{2k} is proportional to $c_2(c_4/c_2)^{k-1}$. We will prove this by induction. Hence let $c_{2j} = \lambda_j c_2(c_4/c_2)^{j-1}$, for all $j \leq k$. We have then using (4.3):

$$\begin{aligned} c_{2k+2} &= \frac{c_4 c_2^2}{2c_2^2} \sum_{j=0}^{k-1} \binom{2k}{2j+1} \lambda_{j+1} \lambda_{k-j} \left(\frac{c_4}{c_2}\right)^j \left(\frac{c_4}{c_2}\right)^{k-j-1} \\ &= c_2 \left(\frac{c_4}{c_2}\right)^k \frac{1}{2} \sum_{j=0}^{k-1} \binom{2k}{2j+1} \lambda_{j+1} \lambda_{k-j}. \end{aligned}$$

Besides we see that numbers λ_k are defined by the recursion:

$$\lambda_{k+1} = \frac{1}{2} \sum_{j=0}^{k-1} \binom{2k}{2j+1} \lambda_{j+1} \lambda_{k-j}.$$

with $\lambda_1 = 1$. Let us consider numbers $T_k = 2^{k-1} \lambda_k$. These numbers satisfy recursion:

$$(4.4) \quad T_{k+1} = \sum_{j=1}^k \binom{2k}{2k-1} T_j T_{k-j+1}.$$

Let us denote by $H(x) = \sum_{k \geq 1} T_k \frac{x^{2k-1}}{(2k-1)!}$. Notice also that $H'(x) = \sum_{k \geq 0} T_{k+1} \frac{x^{2k}}{(2k)!} = 1 + \sum_{k \geq 1} T_{k+1} \frac{x^{2k}}{(2k)!}$. Applying (4.4) we get

$$\begin{aligned} H'(x) &= 1 + \sum_{k \geq 1} \frac{x^{2k}}{(2k)!} \sum_{j=1}^k \binom{2k}{2k-1} T_j T_{k-j+1} = \\ &= 1 + \sum_{j \geq 1} T_j \frac{x^{2j-1}}{(2j-1)!} \sum_{k \geq j} T_{k-j+1} \frac{x^{2k-2j+1}}{(2k-2j+1)!} = \\ &= 1 + H(x)^2. \end{aligned}$$

Since $H(0) = 0$ we see that $H(x) = \tan x$ so indeed numbers T_k are tangent numbers.

Now we know that

$$f'(x) = c_1 + c_2 \sum_{j=1}^{\infty} \frac{x^{2j-1}}{(2j-1)!} T_j \left(\frac{c_4}{2c_2}\right)^{j-1} = c_1 + \sqrt{\frac{2c_2}{c_4}} \tan\left(x \sqrt{\frac{c_4}{2c_2}}\right).$$

Consequently recalling that $\int \tan(ax) dx = -\log \cos(ax)/a$ we arrive at (2.13). Further renumbering that Fourier transform of $1/\cosh(\pi x/(2a))$ is equal to $1/\cosh(at)$ we get (2.14). Keeping in mind that $1/\cosh(bx) = \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} b^{2j} (-1)^j E_{2j}$ where E_{2j} are the so called Euler numbers that have asymptotic $E_{2j} \sim C \sqrt{2j} (4n/(\pi e))^{2n}$ we deduce that even moments of distribution with the density $\frac{1}{2a \cosh(\pi x/(2a))}$ are equal to $b^{2j} (-1)^j E_{2j}$ which by Carleman's criterion leads to identifiability. \square

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